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SUFFICIENT CONDITION THAT TWO LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS SHALL HAVE COMMON INTEGRALS.

By A. B. PIERCE.

As a continuation of the preceding paper I propose to prove that the condition $\triangle=0$ for all values of x is sufficient in order that two linear homogeneous differential equations shall have common integrals. I shall follow a method analogous to that used for the corresponding problem in the Algebra as given by de Comberousse.* Although the problem has been discussed heretofore, by von Escherich and Heffter, I believe this method has not been used before.

Using the theorem that a linear homogeneous differential equation of the nth order has n and only n linearly independent integrals, I prove a sufficient condition expressed in linear homogeneous differential operators and then prove that we can obtain a form of the same character from the above mentioned condition.

Represent a differential operator of order m as follows:

$$A\equiv a_0\frac{d^m}{dx^m}+a_1\frac{d^{m-1}}{dx^{m-1}}+\dots+a_{m-1}\frac{d}{dx}+a_m,$$

and by B, R, S similar operators of order n, p, q, respectively. Further, RAy shall indicate that we operate upon y with A and then upon the result obtained with R; the order of the result will be $m+p$.

LEMMA: *If two linear homogeneous differential operators A and B are such that $RAy+SBy\equiv0$, where $p<n$, $q<m$, then the differential equations $Ay=0$ and $By=0$ have at least one integral in common.*

Let $RAy+BSy\equiv0$; then $m+p=n+q$.

We will carry out our proof for $m=2$, $n=3$.

Represent independent integrals of $Ay=0$ by $y_{a,1}$, $y_{a,2}$, and similarly for B, R, and S. Now considering the differential equations

$$RAy=0 \text{ and } SBy=0,$$

we know from our hypothesis that all the integrals which satisfy one of them must satisfy the other. Then $y_{b,1}$, $y_{b,2}$, $y_{b,3}$ must satisfy $RAy=0$, or $Ay_{b,1}$, $Ay_{b,2}$, $Ay_{b,3}$ satisfy the equation $Ry=0$ where $p\leqq2$.

Since we cannot have more independent integrals of the equation than its order, one of the following possibilities must be true; either

(1) One or more of the expressions Ay_b is zero; or,

(2) One is a linear combination of the other two.

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The first case immediately gives us our theorem. In the second case we have $l_1 Ay_{b1}+l_2 Ay_{b2}+l_3 Ay_{b3}=0$, where l_1, l_2, l_3 are constants. This, however, we may write

$$Al_1 y_{b1}+Al_2 y_{b2}+Al_3 y_{b3}=A(l_1 y_{b1}+l_2 y_{b2}+l_3 y_{b3})=0.$$

That is, $l_1 y_{b1}+l_2 y_{b2}+l_3 y_{b3}$ is an integral of $Ay=0$. Since it is also an integral of $By=0$, by hypothesis, our theorem is proved. By similar reasoning we can prove the following

COROLLARY: *$Ay=0$, and $By=0$, have, at least, $m-q=n-p$ independent integrals in common.*

THEOREM. *$\triangle$ equal zero for all values of x is a sufficient condition that $Ay=0$ and $By=0$ shall have at least one integral in common.*

Here, again, we limit ourselves to the case $m=2$, $n=3$, indicating the generalization for the general case at the close of the proof. Then

$$\triangle \equiv \left|\begin{array}{c|cccc} \alpha_0 & 2\alpha_0'+\alpha_1 & \alpha_0''+2\alpha_1'+\alpha_2 & \alpha_1''+2\alpha_2' & \alpha_2'' \\ \cline{2-5} 0 & \alpha_0 & \alpha_0'+\alpha_1 & \alpha_1'+\alpha_2 & \alpha_2' \\ 0 & 0 & \alpha_0 & \alpha_1 & \alpha_2 \\ 0 & \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ \cline{2-5} \beta_0 & \beta_0'+\beta_1 & \beta_1'+\beta_2 & \beta_2'+\beta_3 & \beta_3' \end{array}\right|$$

We will associate with each column of $\triangle$ a definite derivative of y which we will later use as a factor in connection with the elements of that column.

Starting from the right with the first column, we will use y itself or the zero derivative; with the second column the first derivative, and so on in succession, in general with the kth column, the $(k-1)$st derivative. Let $\triangle$ vanish for all values of x.

We will now divide the proof into two parts and carry each out separately although the method used in the second part includes the first as a special case.

1. All first minors of $\triangle$ not zero.
2. All first minors of $\triangle$ zero.

1. Let the last column be one for which the minors are not all zero. Then multiplying the last column by y and adding to it all the other columns each multiplied by its corresponding derivative, and representing these sums by the letters A and B with subscripts, we obtain the following equation true for all values of x whatever y may be:

$$\left|\begin{array}{ccccc} \alpha_0 & 2\alpha_0'+\alpha_1 & \alpha_0''+2\alpha_1'+\alpha_2 & \alpha_1''+2\alpha_2' & A_2 \\ 0 & \alpha_0 & \alpha_0'+\alpha_1 & \alpha_1'+\alpha_2 & A_1 \\ 0 & 0 & \alpha_0 & \alpha_1 & A_0 \\ 0 & \beta_0 & \beta_1 & \beta_2 & B_0 \\ \beta_0 & \beta_0'+\beta_1 & \beta_1'+\beta_2 & \beta_2'+\beta_3 & B_1 \end{array}\right| \equiv 0.$$

However, $A_0=Ay$, $A_1=\frac{d}{dx}Ay$, $A_2=\frac{d^2}{dx^2}Ay$, and similarly, $B_0=By$, $B_1=\frac{d}{dx}By$.

Expanding according to the elements of the last column, and calling the corresponding cofactors M and N, respectively, with subscripts in the inverse order, we have

$$M_0\frac{d^2}{dx^2}Ay+M_1\frac{d}{dx}Ay+M_2Ay+N_1By+N_0\frac{d}{dx}By\equiv0.$$

Writing $R\equiv M_0\frac{d^2}{dx^2}+M_1\frac{d}{dx}+M_2$, and $S\equiv N_0\frac{d}{dx}+N_1$, we have $RAy+SBy\equiv0$, where $p=2$ and $q=1$.

If the elements of the last columns have their minors zero, we proceed in the same way with some other column for which the hypothesis is satisfied.

2. In this case the matrix obtained by cutting off the first column and the first and last rows of $\triangle$, will vanish, *i. e.*, all the determinants of highest order will vanish.

Assuming that the minor determinants obtained from the elements of the first two columns do not all vanish, we will take those determinants of the matrix which have these two columns for their first two columns, and multiplying each by the derivative corresponding to the last column, we will add them all together. This gives

$$\begin{vmatrix} \alpha_0 & \alpha_0'+\alpha_1 & (\alpha_1'+\alpha_2)\frac{dy}{dx}+\alpha_2'y \\ 0 & \alpha_0 & \alpha_1\frac{dy}{dx}+\alpha_2y \\ \beta_0 & \beta_1 & \beta_2\frac{dy}{dx}+\beta_3y \end{vmatrix}\equiv0.$$

Adding the first two columns, each multiplied by its corresponding derivative to the last column, we may rewrite

$$\begin{vmatrix} \alpha_0 & \alpha_0'+\alpha_1 & A_1 \\ 0 & \alpha_0 & A_0 \\ \beta_0 & \beta_1 & B_0 \end{vmatrix}\equiv0,$$

where the symbols A_0, A_1, B_0 have the same meaning as before. Expanding as before, and writing $R=M_0\frac{d}{dx}+M_1$, $S=N_0$, we have $RAy+SBy\equiv0$.

The next matrix cannot be zero, because then Ay would have all its coefficients zero, which gives a trivial case.

If the first and second columns do not fulfill the conditions imposed, take two which do, and proceed in a similar way. With this, the proof for the case $m=2$, $n=3$ is complete.

For the general case, the proceedure is similar; we have only to consider the possibility of other matrices vanishing. By using the following definition and properties, the general proof is quite evident.

Call first submatrix of $\triangle$ the matrix obtained by cutting off the first column and the first and last rows of $\triangle$; the second submatrix that obtained from the first by the same process; and so on successively.

If any submatrix is zero all the preceding submatrices and $\triangle$ itself are zero. If A and B are not identical, and neither identically zero, then all the submatrices do not vanish. We proceed as in the proof above, with the last one which is zero. From this we easily prove the following

COROLLARY: *If the kth submatrix be the first one, all of whose determinants of highest order do not vanish, then* $Ay=0$, *and* $By=0$, *have k independent integrals in common.*

On this foundation then, we find the number of independent integrals two linear homogeneous equations may have in common, and can also find the equation satisfied by these common integrals.

I add the following numerical cases as illustrations of the lemma:

1. $$Ay=\frac{d^3y}{dx^3}+(x-a)\frac{d^2y}{dx^2}+x(x-a)\frac{dy}{dx}-ax^2y,$$

$$By=\frac{d^2y}{dx^2}+(x-a)\frac{dy}{dx}-axy,$$

$$R=(1-x^2)\frac{d}{dx}+x(3-x^2),$$

$$-S=(1-x^2)\frac{d^2}{dx^2}+x(3-x^2)\frac{d}{dx}-(1-x^2)^2.$$

Then $Ay=0$ and $By=0$, have the common integral $y=e^{ax}$.

2. $$Ay=\frac{d^3y}{dx^3}+x(1+x)\frac{d^2y}{dx^2}+(1+x^2+x^3)\frac{dy}{dx}+x(x^3+2)y,$$

$$By=\frac{d^3y}{dx^3}+2x\frac{d^2y}{dx^2}+(2x^2+1)\frac{dy}{dx}+x(2+x^2)y,$$

$$R=x(x-1)\frac{d}{dx}+(x^3-x^2-2x+1),$$

$$-S=x(x-1)\frac{d}{dx}+(x^4-x^3-2x+1).$$

The common integrals of $Ay=0$ and $By=0$ are integrals of

$$\frac{d^2y}{dx^2}+x\frac{dy}{dx}+x^2y=0.$$

GREIFSWALD, GERMANY, *January, 1903.*